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Module 1 Analysis Basics

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Module 1 Analysis Basics

I) Continuous applications
a) Definitions
Let f be an application defined on an open interval I of $\mathbb{R}$ and with values in $\mathbb{R}$. We say that application f is continuous in $x_0 \in I$ if:
$\forall \epsilon > 0, \exists \eta > 0 / \forall x \in I, |x - x_0| < \eta \Rightarrow |f(x) - f(x_0)| < \epsilon$.
We say that application f is continuous on interval I if it is continuous in any point of interval I.

b) Examples
The polynomial, sinus, cosinus and exponential applications are continuous on $\mathbb{R}$.
An algebraeic fraction, meaning an application of type $f(x) = \frac{P(x)}{Q(x)}$ where P and Q are polynomials is continuous in any real number x such that $Q(x)$ is non zero.
The logarithm application is continuous on $\mathbb{R}^{*+}$.

c) Operations on continuous applications:
Sum, product, quotient
Theorem
Let I be an open interval of $\mathbb{R}$. We denote by f and g two continuous applications on I, then:
Applications $f + g$ and fg are continuous on I.
Application $\frac{f}{g}$ is continuous in any point x such that $g(x)$ is non zero.

Composite of two continuous applications
Theorem
Let I and J be two open intervals of $\mathbb{R}$, f a continuous application from I to J and g a continuous application from J to $\mathbb{R}$, then:
Application $g \circ f$ is continuous on I.

Inverse of a continuous application
Theorem
Let I and J be two intervals of $\mathbb{R}$ and f an bijective application from I to J.
The inverse application of f, denoted by f^{-1} is continuous from J to I.

d) Properties of continuous applications
Intermediate value theorem
Let f be a continuous application on an open interval I, a and b two elements of I.
We have: $\forall \beta \in [f(a), f(b)], \exists \alpha \in [a, b] / f(\alpha) = \beta$.
This means that of application f takes two values, it takes at leat once every value between these two values.

Theorem of bijection

Let I be an interval of $\mathbb{R}$ and f a continuous and strictly monotonous application from I to $\mathbb{R}$. Then the image $f(I)$ of application f is an interval of $\mathbb{R}$ and application f defines a bijection from I to $f(I)$.

The graph representation of application f^{-1} is the symetrix of the graph representation of f with respect to the first bissecting line (meaning the line of equation $y = x$).

Continuous application on a segment :

Theorem

Let I be an open interval of $\mathbb{R}$, $[a, b]$ a segment included in I and f a continuous application from I to $\mathbb{R}$, then:

Application f is bounded on $[a, b]$ and it reaches its bounds.

II) Finite limits

a) Definitions

Let f be an application defined on an open interval I, except in one point $c \in I$, with values in $\mathbb{R}$. We say that application f admits as limit $L \in \mathbb{R}$ when x goes towards c if application $\tilde{f}$ defined on I by: $\forall x \in I, x \neq c, \tilde{f}(x) = f(x)$, $\tilde{f}(c) = L$ is continuous in c.

Application $\tilde{f}$ is called prolongation by continuity of application f.

We denote this by $\underset{x \to c}{Lim} f(x) = L$.

Equivalent definition:

$\forall \epsilon > 0, \exists \eta > 0 / \forall x \in I - \{c\}, |x - c| < \eta \Rightarrow |f(x) - L| < \epsilon.$

One-sided limits:

We denote by a and b two real numbers, $a < b$.

Let f be an application defined on interval $]a, b[$, woth values in $\mathbb{R}$.

We say that application f admits $L \in \mathbb{R}$ as left-sided limit when x goes to b if:

$\forall \epsilon > 0, \exists \eta > 0 / a < b - \eta < x < b \Rightarrow |f(x) - L| < \epsilon$.

We then write $\underset{x \to b, x < b}{\lim} f(x) = L$.

We say that application f admits $L \in \mathbb{R}$ as right-sided limit when x goes to a if

$\forall \epsilon > 0, \exists \eta > 0 / a < x < a + \eta < b \Rightarrow |f(x) - L| < \epsilon$.

We then write $\underset{x \to a, x > b}{\lim} f(x) = L$.

b) Relation between one-sided limits and limit :

Theorem

Let I be an open interval of $\mathbb{R}$, c an element of I and f an application from $I - \{c\}$ to $\mathbb{R}$. Application f has a limit when x goes towards c if and only if f has a right-sided limit and a left-sided limit when x goes towards c and these limits are equal.

We then have : $\underset{x \to c}{\lim} f(x) = \underset{x \to c, x < c}{\lim} f(x) = \underset{x \to c, x > c}{\lim} f(x)$.

c) One-sided continuity:

Let f be a function defined on an interval $[a, b]$.

We define a function g on $]a, b[$ by $f(x) = g(x), \forall x \in]a, b[$.

We say that function f is continuous from the left in b if $\lim\limits_{x \to b, x<b} g(x) = f(b)$ and continuous from the right in a if $\lim\limits_{x \to a, x>a} g(x) = f(a)$
We define in a similar way the continuity from the left and from the right in one point $c \in]a, b[$ and function f is continuous in c if and only if it is continuous from the right and from the left in c.

d) Operations on limits, sum, product, quotient, composition
Theorem : Let f and g be two applications defined on an open interval I, except in one point $c \in I$.
We assume : $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = L'$, then:
$\lim\limits_{x \to c} (f + g)(x) = L + L'$
$\lim\limits_{x \to c} (fg)(x) = LL'$.
If we assume that L' is non zero
$\lim\limits_{x \to c} (\frac{f}{g})(x) = \frac{L}{L'}$.

Theorem
We denote by I and J two intervals from $\mathbb{R}$, by c an element of I, by c' an element of J, by f an application defined from $I - \{c\}$ to J and by g an application defined from $I - \{c'\}$ to J. We assume: $\lim\limits_{c \to c} f(x) = c'$ and $\lim\limits_{x \to c'} g(x) = L$, then:
$\lim\limits_{x \to c} (f \circ g)(x) = L$.

We have the same results for one-sided limits.

III) Infinite limits
a) Definitions
Infinite limits when the variable goes towards a finite value
Let I be an open interval of $\mathbb{R}$, c an element of I, and f an application from $I - \{c\}$ to $\mathbb{R}$. We say that application f goes to $+\infty$ when x goes towards c if $\forall A \in \mathbb{R}^+, \exists \eta > 0 / |x - c| < \eta \Rightarrow f(x) > A$.
We write: $\lim\limits_{x \to c} f(x) = +\infty$.

We define similarly an application which goes to $-\infty$ when x goes towards c:
$\lim\limits_{x \to c} f(x) = -\infty$ if $\forall A \in \mathbb{R}^-, \exists \eta / |x - c| < \eta \Rightarrow f(x) < A$.

One-sided limits:
Let a and b be two real numbers, $a < b$, and f an application from $]a, b[$ to $\mathbb{R}$.
We say that application f goes to $+\infty$ when x goes towards b from below if $\forall A \in \mathbb{R}^+, \exists \eta > 0 / a < b - \eta < x < b \Rightarrow f(x) > A$.
We denote this by: $\lim\limits_{x \to b, x<b} f(x) = +\infty$.
We say that application f goes to $-\infty$ when x goes to a from above if $\forall A \in \mathbb{R}^-, \exists \eta > 0 / a < x < a + \eta < b \Rightarrow f(x) < A$.
We denote this by: $\lim\limits_{x \to a, x>a} f(x) = -\infty$.

Finite limits when the variable goes to infinity

We say that application f goes towards L when x goes to $+\infty$ if
$\forall \epsilon > 0, \exists A \in \mathbb{R}^+ / x > A \Rightarrow |f(x) - L| < \epsilon$.
We write: $\lim_{x \to +\infty} f(x) = L$.
We define similarly:
$\lim_{x \to -\infty} f(x) = L$ by:
$\forall \epsilon > 0, \exists A \in \mathbb{R}^- / x < A \Rightarrow |f(x) - L| < \epsilon$.

Infinite limits when the variable goes to infinity
We say that application f goes to $+\infty$ when x goes to $+\infty$ if
$\forall A \in \mathbb{R}^+, \exists B \in \mathbb{R}^+ / x > B \Rightarrow f(x) > A$.
We write: $\lim_{x \to +\infty} f(x) = +\infty$.
We define similarly $\lim_{x \to +\infty} f(x) = -\infty$ by
$\forall A \in \mathbb{R}^-, \exists B \in \mathbb{R}^+ / x > B \Rightarrow f(x) < A$.
$\lim_{x \to -\infty} f(x) = +\infty$ by
$\forall A \in \mathbb{R}^+, \exists B \in \mathbb{R}^- / x < B \Rightarrow f(x) > A$.
$\lim_{x \to -\infty} f(x) = -\infty$ by
$\forall A \in \mathbb{R}^-, \exists B \in \mathbb{R}^- / x < B \Rightarrow f(x) < A$.

Limits of a sum:
The following table gives, when possible, the limit of the sum of applications f and g depending on the limits of f and g.
The symbol IF (indetermined form) means there is no generic rule.

	L	$+\infty$	$-\infty$
L'	$L + L'$	$+\infty$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	IF
$-\infty$	$-\infty$	IF	$-\infty$

Limit of a product
The following table gives, when possible, the limit of the product of applications f and g depending on the limits of f and g.

	$L > 0$	$L < 0$	0	$+\infty$	$-\infty$
$L > 0$	LL'	LL'	0	$+\infty$	$-\infty$
$L < 0$	LL'	LL'	0	$-\infty$	$+\infty$
0	0	0	0	IF	IF
$+\infty$	$+\infty$	$-\infty$	IF	$+\infty$	$-\infty$
$-\infty$	$-\infty$	$+\infty$	IF	$-\infty$	$+\infty$

Limit of a quotient
The following table gives, when possible, the limit of the quotient $\frac{f}{g}$ depending on the limits of f and g. We say that application f goes towards 0^+ if the limit is zero and the application has positive values. We define similarly an application which goes towards 0 with negative values and the limit is then denoted by 0^-.

	$L>0$	$L<0$	0^+	0^-	$+\infty$	$-\infty$
$L'>0$	LL'	LL'	0	0	$+\infty$	$-\infty$
$L<0$	LL'	LL'	0	0	$-\infty$	$-\infty$
0^+	$+\infty$	$-\infty$	IF	IF	$+\infty$	$-\infty$
0^-	$-\infty$	$+\infty$	IF	IF	$-\infty$	$+\infty$
$+\infty$	0	0	0	0	IF	IF
$-\infty$	0	0	0	0	IF	IF

IV) Differentiable applications

1) Definition

Let f be an application defined on an open interval I of $\mathbb{R}$, with values in $\mathbb{R}$. Let a and x be two elements of I. We call growth rate of f between a and x the quotient $\frac{f(x)-f(a)}{x-a}$.

We say that application f is differentiable in $a$if the growth rate has a finite limit when x goes towards a. The real number $\lim\limits_{x\to a}\frac{f(x)-f(a)}{x-a}$ is called derivative number of application f in a and denoted by $f'(a)$.

If the growth rate has a rigth-sided limit when x goes to a, we say that application f is differentiable from the rigth in a. We define similarly the applications that are differentiable from the left.

Geometric interpretation:

The derivative number $f'(a)$ of a differentiable application in point a is the slope of the tangent line of the graphical representation of application f in the point of coordinates $(a, f(a))$.

Derivative application:

If an application f is differentiable in any point of an interval I, we define an application from I to $\mathbb{R}$, denoted by f' and called derivative application of application f.

2) Sum, produic, quotient and composite of differentiable applications

Let I be an interval of $\mathbb{R}$, f and g two differentiable applications on I.

Application $f+g$ is differentiable and $(f+g)' = f' + g'$

Application fg is differentiable and $(fg)' = f'g + fg'$.

If application g does not take the zero value on interval I, application $\frac{f}{g}$ is differentiable and $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$.

Let I and J be two intervals of $\mathbb{R}$, f a differentiable application from I to J and g an application from J to $\mathbb{R}$.

Application $g \circ f$ is differentiable on I and $(f \circ g)' = g'(f' \circ g)$

Differentiability of the inverse of a bijective application

Let f be a bijective differentiable application defined on an interval I of $\mathbb{R}$ and with values in an interval J. Application f^{-1} is differentiable in a point $y = f(x)$ belonging to J if and only if $f'(x)$ is non zero and we have: $(f^{-1})'(y) = \frac{1}{f'(x)}$.

The derivative application of application f^{-1} is $(f^{-1})' = \frac{1}{f'of^{-1}}$.

3) Derivative and direction of variation of an application
Let f be a differentiable application on an interval I of $\mathbb{R}$.
If application f' is positive on interval I, application f is increasing on I.
If application f' is negative on interval I, application f is decreasing on I.

4) Derivatives of upper orders
Definitions
Let f be a differentiable application on a interval I, with values in $\mathbb{R}$. If application f' is continuous on I, we say that application f is of class C^1 on I.
If application f' is differentiable, we denote by $f"$ its derivative, called second derivative of function f. If the second derivative of f is continuous, we say that f is of class C^2.
We can define in a recursive way the derivative of order n of f, which will be called of class C^n if it has a continuous derivative of order n.
If function f is of class C^n for any natural integer n, we say it is of class C^∞.

5) Rolle's theorem, formula of mean value
Rolle's theorem
Let f be a continuous application on a segment $[a, b]$ of $\mathbb{R}$, differentiable on $]a, b[$, such that $f(a) = f(b)$, then:
There exists a point $c \in]a, b[$ such that $f'(c) = 0$.

Formula of mean value:
Let f be a continuous application on $[a, b]$ and differentiable on $]a, b[$.
There exists $c \in]a, b[$ such that $f(b) - f(a) = b - a)f'(c)$.

V) Taylor-expansion
1) Definition
Let f be an application from $\mathbb{R}$ to $\mathbb{R}$, a Taylor-expansion of f around 0 is an approximation of application f by a polynomial with an upper bound of the error.
$f(x) = a_0 + a_1x + ... + a_nx^n + x^n\epsilon(x)$ with $\lim_{x\to 0} \epsilon(x) = 0$.
We also denote $x^n\epsilon(x)$ by $o(x^n)$.
We say that $a_0 + a_1x + ... + a_nx^n + x^n\epsilon(x)$ is the Taylor-expansion of f of order n around 0.
The Taylor-expansion of order n of an application around 0, when it exists, is unique.

We also define a Taylor-expansion around a real a
$f(x) = a_0 + a_1(x - a) + ... + a_n(x - a)^n + (x - a)^n\epsilon(x)$ with $\lim_{x\to a} \epsilon(x) = 0$.
We also denote $(x - a)^n\epsilon(x)$ by $o((x - a)^n)$ and we have the same definitions and properties as for the Taylor-expansion around 0.

2) Taylor-Youg's formula
Let f be an application of class C^n around a point. We have

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f"(a) + ... + \frac{(x-a)^n}{n!} f^{(n)}(a) + o((x-a)^n)$.
By setting $h = x - a$ we obtain another expression which is often useful
$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f"(a) + ... \frac{h^n}{n!} + o(h^n)$

This infers that an application of class C^n around a has a Taylor-expansion of order n.
Taylor-Young's formula allows to determine the usual Taylor-expansions around 0, that are useful to know, and it is used to determine a Taylor-expansion around a point a which differs from 0.

3) Operations on Taylor-expansions
a) Sum
If two applications f and g have Taylor-expansions of order n around a real number a, application $f + g$ has a Taylor-expansion of order n around a, equal to the sum of the Taylor-expansions of f and g.

b) Product
If two applications f and g have Taylor-expansions of order n around a real number a, of type $P(x) + o(x^n)$ and $Q(x) + o(x^n)$ where P and Q are polynomials, the product fg has a Taylor-expansiion of order n, equal to the part with a degree lower or equal to n of polynomial PQ.

c) Integration
If application f has around 0 a Taylor-expansion of order n, of type $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + o(x^n)$ and if we denote by F the antiderivative which is zero in 0 of application f, we have
$F(x) = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + ... + a_n \frac{x^{n+1}}{n+1} + o(x^{n+1})$

4Taylor-expansions of usual applications around 0
Exponential application and associated applications
$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... + \frac{x^n}{n!} + o(x^n)$
$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + ... + (-1)^n \frac{x^n}{n!} + o(x^n)$
$ch(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + ... + \frac{x^{2n}}{(2n)!} + o(x^{2n})$ (expansion of order $2n$)
$sh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{6} + \frac{x^5}{5!} + ... + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$ (expansion of order $2n+1$)

Trigonométric applications
$cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - .... + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$ (expansion of order $2n$)
$sin(x) = x - \frac{x^3}{6} + \frac{x^5}{5!} - .... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$ (expansion of order $2n+1$)

Application $x \mapsto (1+x)^\alpha$
$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + ... + \frac{\alpha(\alpha-1).....(\alpha-n+1)}{n!} x^n + o(x^n)$
Special case $\alpha = -1$ and associated applications
With the expression of the sum of the terms of a geometric sequence, we have
$\frac{1}{1-x} = 1 + x + x^2 + ... + x^n + o(x^n)$

By changing x to $-x$ we obtain
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + o(x^n)$
By integration we obtain
$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - .... + (-1)^{n+1} x^n + o(x^n)$

VI) Comparison relations and compared growths
Definition :
Let f and g be two applications defined on an interval of type $]a, b]$ where $[b, a[$, a denoting either a real number, either $\pm\infty$.
Applications f and g are equivalent when x goes to a $f(x) = \epsilon(x)g(x)$, with $\lim_{x \to a} \epsilon(x) = 1$.
We write $f \sim g$.

We say that f is negligeable compared to g when x goes to a if $f(x) = \epsilon(x)g(x)$, with $\lim_{x \to a} \epsilon(x) = 0$.
We write $f = o(g)$.

We say that f is dominated by g when x goes to a if $f(x) = \epsilon(x)g(x)$, application ϵ being bounded around a.
We write $f = O(g)$
Compared growths:
Let α and β be two strictly positive real numbers.
$\lim_{x \to 0} L\, x^\alpha (|\ln(x)|)^\beta = 0$ and $\lim_{x \to +\infty} \frac{(\ln(x))^\alpha}{x^\beta} = 0$
We say that polynomial growth dominates logarithmic growth.

$\lim_{x \to -\infty} |x|^\alpha e^{\beta x} = 0$ and $\lim_{x \to +\infty} \frac{e^{\beta x}}{x^\alpha} = 0$.
We say that exponential growth dominates polynomial growth.

VII) Numeric sequences
Definitions
A numeric sequence is an application from $\mathbb{N}$ (or a part of $\mathbb{N}$) with values in $\mathbb{R}$ or $\mathbb{C}$.
We write $n \to u(n)$, or $n \to u_n$, and we denote the sequence (meaning the application) by $(u_n)_{n \in \mathbb{N}}$.
The numeric sequence $(u_n)_{n \in \mathbb{N}}$ is convergent if there exists L, belonging to $\mathbb{R}$ or $\mathbb{C}$ such that :
$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \ / \ n \geqslant N \Longrightarrow |u_n - L| < \epsilon$.
A non convergent sequence is said to be divergent.
We call L the limit of the sequence and we denote it by : $\lim_{n \to +\infty} u_n = L$.

Infinite limits
Let $(u_n)_{n \in \mathbb{N}}$ be a real sequence :
$\lim_{n \to +\infty} u_n = +\infty \Longleftrightarrow \forall A > 0,\ \exists N \in \mathbb{N} \ / \ n > N \Longrightarrow u_n > A$.
The definition is similar for a limit equal to $-\infty$.
A sequence which goes to infinity is divergent. Convergent infers in $\mathbb{R}$ or $\mathbb{C}$.

Main sequence properties :
The limit of a sequence, if it exists, is unique.
Every convergent sequence is bounded.
The set of numeric sequences can be provided with a vector space structure and an inner product.
The sum or the product of two convergent sequences is a convergent sequence, which has as a limit the sum or the product of the original limits.
If a sequence has a non zero finte limit, there exists an index beyond whihch the sequence is never zero. We can then define an inverse sequence, which is convergent and has as a limit the inverse of the original limit.
We have the same result for a quotient of sequences, and expand it in special cases to zero or infinite limits, the cases where we cannot say anything being the indeterminated forms that are similar to the ones seen for applications.
Any real increasing sequence with an upper bound (or decreasing with a lower bound) is convergent.

Subsequences
Let $(u_n)_{n\in\mathbb{N}}$ be a numeric sequence and ϕ be a strictly increasing application from $\mathbb{N}$ to $\mathbb{N}$. We call subsequence of sequence $(u_n)_{n\in\mathbb{N}}$ the sequence $\left(u_{\phi(n)}\right)_{n\in\mathbb{N}}$.
Examples
$(u_{2n})_{n\in\mathbb{N}}$, $(u_{2n+1})_{n\in\mathbb{N}}$, $(u_{n!})_{n\in\mathbb{N}}$, $(u_{2^n})_{n\in\mathbb{N}}$ are subsequences of sequence $(u_n)_{n\in\mathbb{N}}$.

Properties
Every subsequence of a convergent sequence is convergent, with the same limit.

Remark
This result is often handy to prove that a sequence does not converge, for instance :
$u_n = (-1)^n$. The subsequence of terms of even rank is constant and equal to 1, hence converges towards 1 ; the subsequence of terms of odd rank is constant and equal to -1, hence converges towards -1, thererfore the sequence cannot be convergent.

Complementary subsequences :
The p subsequences $\left(u_{\phi_1(n)}\right)_{n\in\mathbb{N}}$, $\left(u_{\phi_2(n)}\right)_{n\in\mathbb{N}}$, ..., $\left(u_{\phi_p(n)}\right)_{n\in\mathbb{N}}$ of sequence $(u_n)_{n\in\mathbb{N}}$ are said to be complementary if :
$\{\phi_1(n), n \in \mathbb{N}\} \cup \{\phi_2(n), n \in \mathbb{N}\} \cup \ldots \cup \{\phi_p(n), n \in \mathbb{N}\} = \mathbb{N}$.

Theorem
If p complementary subsequences of a sequence $(u_n)_{n\in\mathbb{N}}$ converge towards the same limit L (real or complex), then sequence $(u_n)_{n\in\mathbb{N}}$ converges towars L.
If p complementary subsequences of a sequence $(u_n)_{n\in\mathbb{N}}$ go to $+\infty$ (or $-\infty$), then sequence $(u_n)_{n\in\mathbb{N}}$ goes to $+\infty$ (or $-\infty$).

Comparaison relations :
We say that sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are equivalent if $u_n = \epsilon_n v_n$ with $\lim\limits_{n\to+\infty} \epsilon_n = 1$
We write $u_n \sim v_n$.

We say that sequence $(u_n)_{n\in\mathbb{N}}$ is negligeable to sequence $(v_n)_{n\in\mathbb{N}}$ if $u_n = \epsilon_n v_n$ with $\lim\limits_{n\to+\infty} \epsilon_n = 0$
We write $u_n = o(v_n)$.

We say that sequence $(u_n)_{n\in\mathbb{N}}$ is dominated by sequence $(v_n)_{n\in\mathbb{N}}$ if $u_n = \epsilon_n v_n$ and sequence $(\epsilon_n)_{n\in\mathbb{N}}$ is bounded.
We write $u_n = o(v_n)$.

Exercises

1) We denote by I an interval of $\mathbb{R}$ and by f and g two applications defined from I to $\mathbb{R}$ that are continuous in a point $a \in I$.
a) Prove that application $f + g$ is continuous in a.

b) Prove that if application f is continuous in a point $a \in I$, $\exists\eta \in \mathbb{R}^{*+}$ such that application f is bounded on $[a - \eta, a + \eta]$.

c) Prove that application fg is continuous in a.

2) a) Prove that application *sinus* defines a bijection between intervals $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[-1, 1]$. Draw the graphical representations of both applications. The inverse application is denoted by *Arcsin*.

b) Find two intervals I and J such that application *cosinus* defines a bijection from I to J and draw the graphical representations.

3) For any natural integer n greater or equal to 2, we define the application f_n from $\mathbb{R}$ to $\mathbb{R}$ by:
$f_n(x) = x^n - nx + 1$
Prove that equation $f_n(x) = 0$ has a unique solution in interval $[0, 1]$.

4) We denote by f a continuous application from segment $[0, 1]$ into itself.
Prove that application f has a fixed point, meaning there exists $x \in [0, 1]$ such that $f(x) = x$.

5) Prove that if an application f defined on an open interval I except in point $c \in I$ is such that: $\lim\limits_{x\to c, x<c} f(x) = \lim\limits_{x\to c, x>c} f(x) = l$, then f has a limit when x goes towards c, equal to l.

6) Find the following limits:
a) $\lim\limits_{x\to+\infty} \frac{x+1}{x-1}$

b) $\lim_{x \to +\infty} \sqrt{x^2+1} - \sqrt{x^2-1}$.

c) $\lim_{x \to 0} \frac{\cos(x)}{x^2}$.

7) a) Prove that application $Arcsin$ (defined in exercise 2) is differentiable on $]-1,1[$ and calculate its derivative.

b) Prove that application $Arccos$ (defined in exercise 2) is differentiable on $]-1,1[$ and calculate its derivative.

c) Prove that application $Arctan$ (defined in exercise 2) is differentiable on $]-1,1[$ and calculate its derivative.

8) We denote by p and q two real numbers and by n a natural integer greater or equal to 2.
Prove that equation $x^n + px + q = 0$ has at most two roots if n is even and at most three if n is odd.

9) Let f be an application of class C^n on $\mathbb{R}$.
We assume that application f is non zero in $n+1$ distinct real numbers.
Prove that the derivative of order n of f takes the zero value at least once.

10) We denote by P a polynomial.
Prove that equation $P(x) = e^x$ has only a finite number of roots.

11) Let f be a differentiable application from $[0,1]$ to $\mathbb{R}$, such that $f(0) = 0$ and $f(1) = 1$.
Let n be a non zero natural integer, prove that there exists n real numbers $y_0, y_1, \ldots.., y_{n-1}$ belonging to $[0,1]$ such that $\sum_{k=0}^{n-1} f'(y_i) = n$.

12) a) Using the Taylor-expansion of order n of application $x \mapsto \frac{1}{1+x}$, find the Taylor-expansion of order $2n$ around 0 of application $x \mapsto \frac{1}{1+x^2}$.

b) By integrating the previous Taylor-expansion, find the Taylor-expansion of order $2n+1$ around 0 of function $Arctan$.

13) a) Using the Taylor-expansion of application $x \mapsto (1+x)^\alpha$, find the Taylor-expansion of order 3 around 0 of applications
$x \mapsto \sqrt{1+x}$
$x \mapsto \frac{1}{\sqrt{1-x}}$.

b) Infer the Taylor-expansion of order 6 of application
$x \mapsto \frac{1}{\sqrt{1-x^2}}$

c) By integrating, infer from the previous expansion a Taylor-expansion of application
$x \mapsto Arcsin(x)$.

14) Find the Taylor-expansion of order 4 of the following applications:
a) $f(x) = \frac{x}{\sin(x)}$
b) $g(x) = \frac{1}{\cos(x)}$
c) $h(x) = e^{\cos(x)}$

15) By using Taylor-expansions, find the following limits:
a) $\lim\limits_{x \to 0} \frac{x(1+\cos(x))-2\tan(x)}{2x-\sin(x)-\tan(x)}$

b) $\lim\limits_{x \to 0} \frac{\ln(x+\cos(x))-xe^{-x}}{x^3}$

c) $\lim\limits_{x \to +\infty} (\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}+c^{\frac{1}{s}}}{3})^x$ where a, b, c are three strictly positive real numbers.

16) We denote by p a non zero natural integer and we define an application f from $\mathbb{R}^{*+}$ to $\mathbb{R}$ by: $f(x) = (\frac{1^{p/x}+2^{p/x}+.......+p^{p/x}}{p})^x$.
Find $\lim\limits_{x \to +\infty} f(x)$.

17) A hiker walks 12km in one hour.
Prove that there exists an interval of time of a half-hour during which he walks exactly 6km.

18) For any natural integer n greater or equal to 2, we define the function f_n from $[0, 1]$ to $\mathbb{R}$ by: $f_n(x) = x^n - nx + 1$.
a) Show that equation $f_n(x) = 0$ has a unique solution in $[0, 1]$.
We denote this unique solution by x_n.
b) Study the variations of sequence $(x_n)_{n \geq 2}$.
c) Find the limit of sequence $(x_n)_{n \geq 2}$ when n goes to infinity.
d) Find an equivalent of sequence $(x_n)_{n \geq 2}$ when n goes to infinity.

19) We denote by p a fixed natural integer. We define a sequence $(u_n)_{n \in \mathbb{N}^*}$ by:
$u_n = (\frac{1^{p/n}+2^{p/n}+.....+k^{p/n}+.........+p^{p/n}}{p})^n$.
Find the limit of sequence $(u_n)_{n \in \mathbb{N}^*}$.

20) We denote by a a strictly positive real number and by b a real number belonging to interval $]a, 1[$.
We define two real sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ by:
$a_1 = \frac{a+b}{2}$, $b_1 = \sqrt{a_1 b}$ and for any integer n greater or equal to 2 :
$a_n = \frac{a_{n-1}+b_{n-1}}{2}$ and $b_n = \sqrt{a_n b_{n-1}}$.
Finde the limits of sequences $(a_n)_{n \in \mathbb{N}*}$ and $(b_n)_{n \in \mathbb{N}}$.

We can set $\cos(\alpha) = \frac{a}{b}$ and look for the expressions of the sequences depending on $\cos(\frac{\alpha}{2^k})$.